See previous lecture notes for applications.

Topics

- Complex eigenvalues and linear dynamical systems
- Repeated eigenvalues and Jordan Blocks
- Matrix exponential and linear dynamical systems.

These notes do not closely follow the textbook, but draw on material from ALA 8.6, 10.1, 10.3 and 10.4.

Complex Eigenvalues and Linear Dynamical Systems

Let's apply our solution method for linear dynamical systems to $\dot{\underline{x}} = A\underline{x}$, with matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ that we saw last class. Recall that A has eigenvalue/eigenvector pairs:

$$\lambda_1 = i, \ \underline{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \quad \text{and} \quad \lambda_2 = -i, \ \underline{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Even though these have complex entries, we can still use the approach from last class. We write the solution to $\dot{\underline{x}} = A\underline{x}$ as

$$\underline{x}(t) = c_1 e^{\lambda_1 t} \underline{v}_1 + c_2 e^{\lambda_2 t} \underline{v}_2 = c_1 e^{it} \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 1 \\ i \end{bmatrix}, \qquad \text{(SOL)}$$

i.e., $x(t)$ is a linear combination of the two solutions

$$\underline{x}_1(t) = c_1 e^{it} \begin{bmatrix} 1 \\ -i \end{bmatrix} \quad \text{and} \quad \underline{x}_2(t) = c_2 e^{-it} \begin{bmatrix} 1 \\ i \end{bmatrix}, \qquad \text{(BASE)}$$

and then solve for c_1 and c_2 to ensure compatibility with the initial condition $x(0)$. In general c_1 and c_2 will also be complex numbers, and $x(t)$ will take complex values. This is mathematically correct, and indeed one can study dynamical systems evolving over complex numbers.

However, in this class, and in most engineering applications, we are interested in real solutions to $\dot{x} = Ax$. If we know we want real solutions, it might make sense to try to find different "base" solutions than $x_1(t)$ and $x_2(t)$ that still span all possible solutions to $\dot{x} = Ax$. Our key tool for accomplishing this is Eulers formula, which states that for any $t \in \mathbb{R}$,

$$e^{it} = \cos t + i \sin t \qquad \text{(EUL)}.$$

We apply (EUL) to (BASE), and obtain (after simplifying):

$$\underline{x}_1(t) = e^{it} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + i \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$$

and $$\underline{x}_2(t) = e^{-it} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} - i \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

We've made some progress, in that $x_1(t)$ and $x_2(t)$ are now in the "standard" complex number form $a+ib$, and that $x_1(t) = \bar{x}_2(t)$, i.e., they are clearly complex conjugates of each other. We use this observation strategically to define two new "base" solutions:

$$\hat{\underline{x}}_1(t) = \frac{1}{2}\left(\underline{x}_1(t) + \underline{x}_2(t)\right) = \frac{1}{2}\left(\underline{x}_1(t) + \bar{\underline{x}}_1(t)\right) = \begin{bmatrix}\cos t\\ \sin t\end{bmatrix} \quad \left(= \mathrm{Re}\{\underline{x}_1(t)\}\right)$$

$$\hat{\underline{x}}_2(t) = \frac{1}{2i}\left(\underline{x}_1(t) - \underline{x}_2(t)\right) = \frac{1}{2i}\left(\underline{x}_1(t) - \bar{\underline{x}}_1(t)\right) = \begin{bmatrix}\sin t\\ -\cos t\end{bmatrix} \quad \left(= \mathrm{Im}\{\underline{x}_1(t)\}\right)$$

We note that since $\hat{\underline{x}}_1(t)$ and $\hat{\underline{x}}_2(t)$ are linear combinations of $\underline{x}_1(t)$ and $\underline{x}_2(t)$, they are valid solutions to $\dot{\underline{x}} = A\underline{x}$. Furthermore, since $\hat{\underline{x}}_1(t)$ and $\hat{\underline{x}}_2(t)$ are linearly independent (i.e., $c_1\hat{\underline{x}}_1(t) + c_2\hat{\underline{x}}_2(t) = 0$ for all $t \Leftrightarrow c_1 = c_2 = 0$), they form a basis for the solution set to $\dot{\underline{x}} = A\underline{x}$. Therefore, we can rewrite (SOL) as

$$\underline{x}(t) = c_1\begin{bmatrix}\cos t\\ \sin t\end{bmatrix} + c_2\begin{bmatrix}\sin t\\ -\cos t\end{bmatrix},$$

and then solve for c_1 and c_2 using $\underline{x}(0)$. If $\underline{x}(0) \in \mathbb{R}^2$, i.e., if the initial condition $\underline{x}(0)$ is real, then c_1 and c_2 will be too. For example, suppose $\underline{x}(0) = \begin{bmatrix}a\\ b\end{bmatrix}$, with $a, b \in \mathbb{R}$. Then:

$$\underline{x}(0) = c_1\begin{bmatrix}1\\ 0\end{bmatrix} + c_2\begin{bmatrix}0\\ -1\end{bmatrix} = \begin{bmatrix}c_1\\ -c_2\end{bmatrix} = \begin{bmatrix}a\\ b\end{bmatrix} \Rightarrow c_1 = a,\ c_2 = -b,$$

and $$\underline{x}(t) = \begin{bmatrix}a\cos t - b\sin t\\ a\sin t + b\cos t\end{bmatrix} = \begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = R(t)\,\underline{x}(0),$$

i.e., the solution $\underline{x}(t)$ corresponds to the initial condition $\underline{x}(0)$ being <u>rotated</u> in a counterclockwise direction at a frequency of 1 rad/s.

The key steps in the above procedure were:

① Apply Euler's formula to rewrite the basic solutions as

$$\underline{x}_1(t) = \mathrm{Re}\{\underline{x}_1(t)\} + i\,\mathrm{Im}\{\underline{x}_1(t)\},\quad \underline{x}_2(t) = \bar{\underline{x}}_1(t) = \mathrm{Re}\{\underline{x}_1(t)\} - i\,\mathrm{Im}\{\underline{x}_1(t)\}$$

② Define new basic solutions by setting:

$$\hat{\underline{x}}_1(t) = \mathrm{Re}\{\underline{x}_1(t)\} \quad \text{and} \quad \hat{\underline{x}}_2(t) = \mathrm{Im}\{\underline{x}_2(t)\}.$$

It turns out that this approach is completely general, and can be applied whenever you encounter complex eigenvalue/vectors (which always appear as complex conjugate pairs).

Example: Consider the linear dynamical system $\dot{\underline{x}} = A\underline{x}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 2 & 2 & -1 \end{bmatrix} \quad \text{and} \quad \underline{x}(0) = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}.$$

Using the formula for the determinant of a 3x3 matrix (you don't need to memorize this), we can compute the following eigenvalue/vector pairs:

$$\lambda_1 = -1 \qquad \lambda_2 = 1 + 2i \qquad \lambda_3 = 1 - 2i$$
$$\underline{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \qquad \underline{v}_2 = \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix} \qquad \underline{v}_3 = \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}$$

and obtain the corresponding eigensolutions:

$$\underline{x}_1(t) = e^{-t}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{x}_2(t) = e^{(1+2i)t}\begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}, \quad \underline{x}_3(t) = e^{(1-2i)t}\begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}.$$

Let's apply Euler's formula to $\underline{x}_2(t)$ (remember that $e^{(1+2i)t} = e^t e^{i(2t)}$):

$$\underline{x}_2(t) = e^{(1+2i)t}\begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix} = e^t(\cos 2t + i\sin 2t)\begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} e^t\cos 2t \\ -e^t\sin 2t \\ e^t\cos 2t \end{bmatrix} + i\begin{bmatrix} e^t\sin 2t \\ e^t\cos 2t \\ e^t\sin 2t \end{bmatrix}.$$

This means another set of real eigensolutions to $\dot{\underline{x}} = A\underline{x}$ is

$$\underline{x}_1(t) = e^{-t}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \hat{\underline{x}}_2(t) = \operatorname{Re}\{\underline{x}_2(t)\} = e^t\begin{bmatrix} \cos 2t \\ -\sin 2t \\ \cos 2t \end{bmatrix},$$

$$\hat{\underline{x}}_3(t) = \operatorname{Im}\{\underline{x}_2(t)\} = e^t\begin{bmatrix} \sin 2t \\ \cos 2t \\ \sin 2t \end{bmatrix},$$

and a general solution can be written as

$$\underline{x}(t) = c_1\underline{x}_1(t) + c_2\hat{\underline{x}}_2(t) + c_3\hat{\underline{x}}_3(t) = \begin{bmatrix} -c_1e^{-t} + c_2e^t\cos 2t + c_3e^t\sin 2t \\ c_1e^{-t} - c_2e^t\sin 2t + c_3e^t\cos 2t \\ c_1e^{-t} + c_2e^t\cos 2t + c_3e^t\sin 2t \end{bmatrix}$$

We compute our constants c_1, c_2, c_3 by solving

$$\underline{x}(0) = \begin{bmatrix} -c_1 + c_2 \\ c_1 + c_3 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \Rightarrow \begin{array}{l} c_1 = -2 \\ c_2 = 0 \\ c_3 = 1 \end{array},$$

thus obtaining the specific solution to our original initial value problem as:

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} + e^t \sin 2t \\ -2e^{-t} + e^t \cos 2t \\ -2e^{-t} + e^t \sin 2t \end{bmatrix}.$$

Repeated Eigenvalues, Jordan Forms, and Linear Dynamical Systems

Let's revisit the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ we saw last class. This matrix has an eigenvalue $\lambda = 2$ of algebraic multiplicity 2 ($\det(A - \lambda I) = (\lambda - 2)^2 = 0$ $\Leftrightarrow \lambda_1 = 2$ and $\lambda_2 = 2$) but geometric multiplicity 1, i.e., only one linearly independent eigenvector

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

exists. How can we solve $\dot{\underline{x}} = A\underline{x}$ in this case? Taking the approach that we've seen so far, we would write a candidate solution as

$$\underline{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

But this won't work! What if $\underline{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$? There is no $c_1 \in \mathbb{R}$ such that $\underline{x}(0) = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Does this mean no solution to $\dot{\underline{x}} = A\underline{x}$ exists? This would be deeply unsettling! The issue here is that we are "missing" an eigenvector. To remedy this, we'll introduce the idea of a generalized eigenvector. We will only consider 2×2 matrices, in which case a generalized eigenvector $\underline{v}_2$ for an eigenvalue λ with eigenvector $\underline{v}_1$ is given by the solution to the linear system:

$$(A - \lambda I)\underline{v}_2 = \underline{v}_1. \quad (*)$$

For our example, we compute $\underline{v}_2$ by solving:

$$\left(\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix} - 2\begin{bmatrix}1 & \\ & 1\end{bmatrix}\right)\begin{bmatrix}v_{21}\\ v_{22}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_{21}\\ v_{22}\end{bmatrix} = \begin{bmatrix}v_{22}\\ 0\end{bmatrix} = \underbrace{\begin{bmatrix}1\\ 0\end{bmatrix}}_{\underline{v}_1}$$

$$A \quad -\lambda \quad I$$

$\Rightarrow v_{22} = 1$ and v_{21} is free. We set $v_{21} = 0$ and find

$\underline{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ (any choice for v_{21} would work, this is just a convenient choice).

Now, how can we construct a solution using $\underline{v}_2$? If we try the strategy we used for eigenvalue/vector pairs, things do not quite work out:

If $\underline{x}_2(t) = e^{2t}\underline{v}_2$ then $\dot{\underline{x}}_2(t) = 2e^{2t}\underline{v}_2 = 2\underline{x}_2$

but $A\underline{x}_2(t) = A(e^{2t}\underline{v}_2) \underset{(i)}{=} e^{2t}(2\underline{v}_2 + \underline{v}_1) = 2\underline{x}_2 + \underline{v}_1 e^{2t}$,

where we used the fact that the generalized eigenvector $\underline{v}_2$ satisfies

$$A\underline{v}_2 = \lambda\underline{v}_2 + \underline{v}_1,$$

which is obtained by rearranging (*). So we'll have to try something else. Let's see if

$$\underline{x}_2(t) = e^{2t}\underline{v}_2 + te^{2t}\underline{v}_1$$

does better. This guess is made because we need to find a way to have $e^{2t}\underline{v}_1$ appear in $\dot{\underline{x}}$.

First we compute $\dot{\underline{x}}_2 = 2e^{2t}\underline{v}_2 + e^{2t}\underline{v}_1 + 2te^{2t}\underline{v}_1$

$$= 2(e^{2t}\underline{v}_2 + te^{2t}\underline{v}_1) + e^{2t}\underline{v}_1$$

$$= 2\underline{x}_2 + e^{2t}\underline{v}_1$$

This looks promising! Now let's check

$$A\underline{x}_2(t) = A(e^{2t}\underline{v}_2 + te^{2t}\underline{v}_1) = 2e^{2t}\underline{v}_2 + e^{2t}\underline{v}_1 + 2te^{2t}\underline{v}_1$$

$$= 2(e^{2t}\underline{v}_2 + te^{2t}\underline{v}_1) + e^{2t}\underline{v}_1$$

$$= 2\underline{x}_2 + e^{2t}\underline{v}_1.$$

Success! We therefore can write solutions to our initial value problem as linear combinations of

$$\underline{x}_1(t) = e^{2t}\underline{v_1} \quad \text{and} \quad \underline{x}_2(t) = e^{2t}\underline{v_2} + te^{2t}\underline{v_1},$$

i.e.,

$$\underline{x}(t) = (c_1 + c_2 t)e^{2t}\underline{v_1} + c_2 e^{2t}\underline{v_2}.$$

Let's check if we can find c_1 and c_2 so that $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

$$x(0) = c_1\underline{v_1} + c_2\underline{v_2} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 0 \\ c_2 = 1 \end{matrix}$$

and $x(t) = \begin{bmatrix} te^{2t} \\ e^{2t} \end{bmatrix}$ is the solution to our initial value problem.

2x2 Jordan Blocks

In the complete matrix setting, we saw that we could diagonalize the matrix A using a similarity transformation defined by the eigenvectors of A, i.e., for $V = [\underline{v_1}\ v_2 \cdots v_n]$, we have that

$$\Lambda = V^{-1}AV, \text{ or equivalently, } A = V\Lambda V^{-1}, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \dots \lambda_n).$$

We saw that this was very useful when solving systems of linear equations.

In the case of incomplete matrices, similarity transformations defined in terms of generalized eigenvectors and Jordan blocks play an analogous role.

For example, consider the matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$. This matrix has a repeated eigenvalue at $\lambda = 2$, and one eigenvector $\underline{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We therefore compute the generalized eigenvector by solving $(A - \lambda I)\underline{v_2} = \underline{v_1}$:

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow -v_{21} + v_{22} = 1$$

One solution is $\underline{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We construct our similarity transformation as before, and set $V = [\underline{v_1}\ v_2] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, and compute $V^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

Let's see what happens if we compute $V^{-1}AV$. In the complete case, this would give us a diagonal matrix. In this case, we get

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},$$

which we'll recognize as our previous example! It turns out that all 2×2 matrices with $\lambda=2$ having algebraic multiplicity 2 and geometric multiplicity 1 are similar to the Jordan Block

$$J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},$$

and this similarity transformation is defined by $V=[\underline{v}_1 \ \underline{v}_2]$ composed of the eigenvector $\underline{v}_1$ and generalized eigenvector $\underline{v}_2$ of the original matrix.

We can generalize this idea to any 2×2 matrix with only one eigenvector:

Theorem: Let $A\in\mathbb{R}^{2\times2}$ have eigenvalue λ with algebraic multiplicity 2 and geometric multiplicity 1. Let $\underline{v}_1$ and $\underline{v}_2$ satisfy:

$$(A-\lambda I)\underline{v}_1 = 0 \quad \text{and} \quad (A-\lambda I)\underline{v}_2 = \underline{v}_1 .$$

Then $A = VJ_\lambda V^{-1}$, where $V=[\underline{v}_1 \ \underline{v}_2]$ and J_λ is the Jordan Block

$$J_\lambda = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

Using this theorem, we can conclude, much in the same way we did for diagonalizable A, that if $A = VJ_\lambda V^{-1}$, then

$$\underline{x}(t) = (c_1 + c_2 t)e^{\lambda t}\underline{v}_1 + c_2 e^{\lambda t}\underline{v}_2$$

is a general solution to $\dot{\underline{x}} = A\underline{x}$.

NOTE: This is a very specific instantiation of the Jordan Canonical Form of a matrix. You will learn more about the Jordan Canonical Form and its implications on differential equations in ESE 2100. For those interested in the fully general theorem statement, see ALA 8.6, Theorem 8.51.

Matrix Exponential

We've seen four cases for eigenvalues/eigenvectors and their relationship to solutions of initial value problems defined by $\dot{\underline{x}} = A\underline{x}$ and $\underline{x}(0)$ given:

1) real distinct eigenvalues, solved by diagonalization;
2) real repeated eigenvalues w/ algebraic multiplicity = geometric multiplicity, also solved by diagonalization;
3) complex distinct eigenvalues, solved by diagonalization and applying Euler's formula to define real-valued eigenfunctions;
4) repeated eigenvalues with algebraic multiplicity > geometric multiplicity, solved by Jordan decomposition using generalized eigenvectors.

While correct, the fact that there are four different cases we need to consider is somewhat unsatisfying. In this section, we show that by appropriately defining a matrix exponential, we can provide a unified treatment of all the aforementioned settings.

We start by recalling the power series definition for the scalar exponential e^x, for $x \in \mathbb{R}$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad \text{(PS)}$$

where we recall that $k! = 1 \cdot 2 \cdots (k-1) \cdot k$. We know that for the scalar initial value problem $\dot{x} = ax$, the solution is $x(t) = e^{at} x(0)$, where e^{at} can be computed via (PS) by setting $x = at$.

Wouldn't it be cool if we could do something similar for the vector valued initial value problem defined by $\dot{\underline{x}} = A\underline{x}$? Does there exist a function, call it e^{At}, so that $\underline{x}(t) = e^{At}\underline{x}(0)$? How would we even begin to define such a thing?

Let's do the "obvious" thing, and start with the definition (PS), and replace the scalar x with a matrix X to obtain the matrix exponential of X:

$$e^X = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{X^k}{k!}, \qquad \text{(MPS)}$$

Although we won't prove it, it can be shown that (MPS) converges for any X, so this is a well defined object. Does (MPS) help with solving $\dot{\underline{x}} = A\underline{x}$? Let's try the test solution $\underline{x}(t) = e^{At}\underline{x}(0)$ — this is exactly what we did for the scalar setting, but we replace e^{at} with e^{At}. Is this a solution to $\dot{\underline{x}} = A\underline{x}$?

First, we compute $A\underline{x}(t) = Ae^{At}\underline{x}(0)$. Next, we need to compute $\frac{d}{dt} e^{At}\underline{x}(0)$.

But how do we do this? We will rely on (MPS):

$$\frac{d}{dt} e^{At}\underline{x}(0) = \frac{d}{dt}\left(I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots\right)$$

$$= \frac{d}{dt} I + \frac{d}{dt} At + \frac{d}{dt}\frac{A^2t^2}{2!} + \frac{d}{dt}\frac{A^3t^3}{3!} + \cdots$$

$$= 0 + A + \frac{\cancel{2}A^2t}{\cancel{2}\cdot 1} + \frac{A^3\cancel{3}t^2}{\cancel{3}\cdot 2\cdot 1} + \cdots$$

$$= A + A^2t + \frac{A^3t^2}{2!} + \frac{A^4t^3}{3!} + \cdots$$

$$= A\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots\right)$$

$$= A e^{At}\underline{x}(0).$$

This worked, and we have found a general solution to $\dot{\underline{x}} = A\underline{x}$ defined in terms of the matrix exponential!

<u>Theorem</u>: Consider the initial value problem $\dot{\underline{x}} = A\underline{x}$, with $\underline{x}(0)$ specified. Its solution is given by $\underline{x}(t) = e^{At}\underline{x}(0)$, where e^{At} is defined according to the matrix power series (MPS)

This is very satisfying, as now our scalar and vector-valued problems have similar looking solutions defined in terms of appropriate exponential functions. The only thing that remains is to compute e^{At}! How do we do this? This is where all of the work we've done on diagonalization and Jordan forms really pays off!

<u>Case 1: Real eigenvalues, diagonalizable A</u>

Suppose that $A \in \mathbb{R}^{n\times n}$, and has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with corresponding linearly independent eigenvectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$. Then we can write

$$A = V\Lambda V^{-1}, \text{ for } V = [\underline{v}_1\ \underline{v}_2 \cdots \underline{v}_n] \text{ and } \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

To compute e^{At}, we need to compute powers $(At)^k$. Let's work a few of these out using $A = V\Lambda V^{-1}$:

$$(At)^0 = I, \quad At = V\Lambda V^{-1}t, \quad A^2t^2 = (V\Lambda V^{-1})(V\Lambda V^{-1})t^2 = V\Lambda^2V^{-1}t^2, \quad A^3t^2 = (V\Lambda V^{-1})A^2t^3 = (V\Lambda V^{-1})(V\Lambda^2V^{-1})t^3 = V\Lambda^3V^{-1}t^3$$

There is a pattern: $(At)^k = V\Lambda^k V^{-1} t^k$. This is nice, since computing powers of diagonal matrices is easy:

$$\Lambda^k = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}.$$

Let's plug these expressions into (MPS):

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$= VV^{-1} + V\Lambda V^{-1}t + \frac{V\Lambda^2V^{-1}t^2}{2!} + \frac{V\Lambda^3V^{-1}t^3}{3!} + \cdots$$

$$= V\left(I + \Lambda t + \frac{\Lambda^2t^2}{2!} + \frac{\Lambda^3t^3}{2!} + \cdots\right)V^{-1} \qquad (\text{factor out } V(\ )V^{-1})$$

$$= V\left(\operatorname{diag}\left(1+\lambda_1t+\frac{\lambda_1^2t^2}{2!}+\frac{\lambda_1^3t^3}{3!}, \ldots, 1+\lambda_nt+\frac{\lambda_n^2t^2}{2!}+\frac{\lambda_n^3t^3}{3!}\right)\right)V^{-1}$$

$$= V\begin{bmatrix} e^{\lambda_1t} & & & \\ & e^{\lambda_2t} & & \\ & & \ddots & \\ & & & e^{\lambda_nt} \end{bmatrix}V^{-1} \qquad \left(\text{we recognize } 1+\lambda_1t+\frac{\lambda_1^2t^2}{2!}+\cdots \text{ as (PS)}\right)$$

That's very nice! We diagonalize A, then exponentiate its eigenvalues to compute e^{At}. Let's plug this back in to $\underline{x}(t) = e^{At}\underline{x}(0)$:

$$\underline{x}(t) = V\begin{bmatrix} e^{\lambda_1t} & & \\ & \ddots & \\ & & e^{\lambda_nt} \end{bmatrix}V^{-1}\underline{x}(0).$$

Now, if we let $\underline{c} = V^{-1}\underline{x}(0)$, we can write

$$\underline{x}(t) = [\underline{v}_1 \cdots \underline{v}_n]\begin{bmatrix} e^{\lambda_1t} & & \\ & \ddots & \\ & & e^{\lambda_nt} \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1e^{\lambda_1t}\underline{v}_1 + \cdots + c_ne^{\lambda_nt}\underline{v}_n,$$

recovering our previous solution, with the exact formula $\underline{c} = V^{-1}\underline{x}(0)$ we saw previously for the coefficients $c_1, \ldots, c_n$!

Case 2: Imaginary eigenvalues

We focus on the 2x2 case with $A = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} = \omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. In this case, we will compute the power series directly.

$$A = \omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \omega J, \quad A^2 = \omega^2 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \omega^2 J^2, \quad A^3 = \omega^3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \omega^3 J^3, \quad A^4 = \omega^4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \omega^4 J^4$$

$$A^5 = \omega^5 J^5 = \omega^5 J, \quad A^6 = \omega^6 J^6 = J^2, \quad A^7 = \omega^7 J^7 = \omega^7 J^3, \quad A^8 = \omega^8 J^8 = \omega^8 J^4,$$

etc. So putting this together in computing e^{At}, we get:

$$e^{At} = \begin{bmatrix} 1 - \frac{1}{2!}t^2\omega^2 + \cdots, & t\omega - \frac{1}{3!}t^3\omega^3 + \cdots \\ -t\omega + \frac{1}{3!}t^3\omega^3 + \cdots, & 1 - \frac{1}{2!}t^2\omega^2 + \cdots \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix},$$

where we used the power series for $\sin\omega t$ and $\cos\omega t$ in the last equality. As expected, the matrix $A = \omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has a matrix exponential which defines a rotation, at rate ω, so that

$$x(t) = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} x(0).$$

Case 3: Complex Eigenvalues

Let's generalize our previous example to $A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$. The matrix A has complex conjugate eigenvalues $\lambda_1 = \sigma + i\omega$ and $\lambda_2 = \sigma - i\omega$. We will again compute the power series directly. To do so, we will use the following very useful fact:

Fact: $e^{A+B} = e^A e^B$ if and only if $AB = BA$, that is, if and only if A and B commute.

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We will strategically use this fact. First, defining $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ we note we can write $A = \sigma I + \omega J$. Importantly, σI and ωJ commute as $(\sigma I)(\omega J) = (\omega J)(\sigma I) = \omega\sigma J$.

Therefore, $e^{At} = e^{(\sigma I + \omega J)t} = e^{\sigma I t} e^{\omega J t} = \begin{bmatrix} e^{\sigma t} & \\ & e^{\sigma t} \end{bmatrix} \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} = e^{\sigma t} \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}.$

Case 4: Jordan Block

Assume $A = V \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} V^{-1}$, for $V = [\underline{v}_1 \ \ \underline{v}_2]$ an eigenvector and generalized eigenvector of A.

Then following the same argument as case 1, we have that $e^{At} = V e^{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}t} V^{-1}$.

To compute $e^{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}t}$, we note $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}t = t\lambda I + t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and that these two terms commute. Hence: $e^{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}t} = e^{\begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix}t} e^{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}t}$. We note that

$$e^{\begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix}t} = \begin{bmatrix} e^{\lambda t} & \\ & e^{\lambda t} \end{bmatrix} \quad \text{and} \quad e^{\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \quad \text{(higher powers} = 0)$$

$$= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Allowing us to conclude that $e^{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}t} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$, and that

$$\underline{x}(t) = e^{At}\underline{x}(0) = [\underline{v}_1 \ \ \underline{v}_2] \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} V^{-1}\underline{x}(0), \quad \text{and letting } \underline{c} = V^{-1}\underline{x}(0)$$

$$= [\underline{v}_1 \ \ \underline{v}_2] \begin{bmatrix} c_1 e^{\lambda t} + c_2 te^{\lambda t} \\ c_2 e^{\lambda t} \end{bmatrix} = (c_1 e^{\lambda t} + c_2 te^{\lambda t})\underline{v}_1 + c_2 e^{\lambda t} \underline{v}_2,$$

which we recognize from our previous section on Jordan Blocks.